Continuum and Time: Weyl after Heidegger

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So wie die Dinge jetzt stehen, muß aber konstatiert werden: Die große Aufgabe, welche seit der Pythagoreischen Entdeckung des Irrationalen gestellt ist, das uns (namentlich in der fließenden Zeit und der Bewegung) unmittelbar anschaulich gegebene *Stetige* nach seinem in 'exakten' Erkenntnissen formulierbaren Gehalt als Gesamtheit diskreter 'Stadien' mathematisch zu erfassen, dieses Problem ist trotz Dedekind, Cantor und Weierstraß heute so ungelöst wie je. Systeme mehr oder minder willkürlicher Festsetzungen können uns da nicht weiter helfen (mögen sie noch so 'denkökonomisch' und 'fruchtbar' sein); wir müssen versuchen, zu einer auf Sacheinsicht gegründeten Lösung zu gelangen.

Hermann Weyl *Das Kontinuum* 1918 p. 16.

As things now stand, however, it must be stated: the great task posed since the Pythagorean discovery of the irrational, to mathematically grasp the *continuity* immediately given to us intuitively (especially in flowing time and movement) according to its content as formulable in 'exact' knowledge as a totality of discrete 'stages' — this problem is today as unsolved as it ever was, despite Dedekind, Cantor and Weierstraß. Systems of more or less arbitrary postulations cannot help us further here (no matter how 'economical in thought' and 'fertile' they may be); we must try to attain a solution based on insight into things.

Continuum and Time: Weyl after Heidegger[1]

0. Abstract

In a section of his WS 1924/25 *Sophistes* lectures, while discussing Aristotle's ontology of continuity, Heidegger refers to Hermann Weyl's work on the continuum in which also Aristotle serves as a source. Heidegger expresses the hope that physicists one day would learn something about movement from Aristotle, a hope that remains unfulfilled to the present day. In recent years, nevertheless, there has been interest in Weyl's thinking on the continuum in Anglophone articles published by mathematicians and philosophers of mathematics. Weyl himself draws on Husserl's subjectivist phenomenology of movement and time as a fundamental intuition of the continuum. His later commentators attempt to exit the inside of consciousness to reach an "intersubjective objectivity" (Feferman). Such intersubjectivity, however, proves itself to be misconceived for attaining anything like an adequate phenomenological understanding of the continuum, movement and time. Against the foil of Dedekind's famous 'cut', an alternative is presented that questions the 'existence' of the real number continuum and shows also that an intuition of linear clock-time is insufficient. Rather, the three-dimensionality of the time-clearing must be brought into play.

1. Interest in Weyl's work on the continuum

In the past couple of decades, Weyl's richly thoughtful work on the continuum has drawn the interest of some Anglophone mathematicians working in the area of the foundations of mathematics, most notably Solomon Feferman and John R. Bell (see the list of references). Weyl's seminal work, *Das Kontinuum* from 1918, is foundational in a deep sense insofar as it hazards to venture to make a connection between

[1] Many thanks to Val Dusek for his resonance to this study.

mathematical conceptions of continuity and phenomena of continuity as experienced in everyday life. He does so by drawing on Husserl's subjectivist phenomenology in its application to the phenomenon of (inner) *time*. It is precisely the problem of how phenomena of continuity are translated into mathematical conceptions of the continuum of real numbers that interests both Weyl and me. It is not arbitrary that the phenomenon of time comes to be focused upon in its relation to the mathematical continuum, for the mathematization of time as a continuous real variable has played a pre-eminent role in the 'phenomenally effective' success of the physical sciences since Descartes and Newton.

Philosophical conceptions of continuity, however, go back to antiquity, in particular, to Aristotle's ontology of continuity as enunciated in his *Physics*. Heidegger was aware of Weyl's work on the foundations of mathematics and physics, explicitly referring to it in his *Sophistes* lectures in WS 1924/25 and making the connection between the continuum and movement:

> Die Frage des continuum ist in der heutigen Mathematik wieder aufgerollt. Man kommt auf aristotelische Gedanken zurück, sofern man verstehen lernt, daß das continuum nicht analytisch auflösbar ist, sondern daß man dahin kommen muß, es als etwas Vorgegebenes zu verstehen, vor der Frage nach einer analytischen Durchdringung. Die Arbeit in dieser Richtung hat der Mathematiker Hermann Weyl geleistet und sie vor allem für die Grundprobleme der mathematischen Physik fruchtbar gemacht. [...] Auf dieses Verständnis des continuum kam er im Zusammenhang mit der Relativitätstheorie [...] Aus diesem Entwicklungsgang kann man erhoffen, daß die Physiker mit der Zeit vielleicht dazu kommen, mit Hilfe der Philosophie zu verstehen, was Aristoteles unter Bewegung verstanden hat. (Heidegger GA19:117f)

> The question regarding the continuum is again being unfolded in today's mathematics. One comes back to Aristotelean thoughts insofar as one learns to understand that the continuum cannot be resolved analytically, but that one must get to the point of understanding it as something pregiven, prior to the question concerning an analytical penetration. The work in this direction has been performed by the mathematician Hermann Weyl (*Raum - Zeit - Materie: Vorlesungen über allgemeine Relativitätstheorie* Berlin 1918) and has been made fecund for the foundational problems of mathematical physics. He came to this understanding of the continuum in connection with the relativity theory [...] From this course of development one can hope that, in time, physicists will

> perhaps come to understand, with the help of philosophy, what Aristotle understood by movement [...]

Although Heidegger is famous for his magnum opus, *Sein und Zeit* (1927), and this work focuses specifically on a fundamental phenomenon crucial to mathematized physics, namely, *time*, as far as I know, over a period now approaching a century, Heidegger's philosophical recasting of the phenomenon of time has not drawn the attention of any mathematicians or analytic philosophers of science working on problems in the foundations of mathematics or the mathematico-physical sciences. Given the patent animosity displayed by analytic philosophy toward Heidegger's phenomenology, this hardly comes as a surprise. Truth must be logical, rational, they claim with innuendo, but what about the irrational real numbers?

The closest mathematicians come to phenomenology, when they eschew Frege's formalism, is via Weyl's drawing upon it in 1918 in order to grapple with the connection between the phenomenon of continuity of movement and the mathematical conception of continuum. Therefore, to clear the ground, it is instructive to first look at what mathematicians make of Weyl's attempt at establishing this connection. Pertinent here in particular are Bell (2000) and Feferman (2000, 2009). By moving backwards along this path, I surmise, it will be possible to finally come "zu den Sachen selbst", i.e. to the issues themselves, namely, of time, continuity and their mathematization. Feferman (2009) cites Weyl's starting-point in *Das Kontinuum* as the following:

> Bleiben wir, um das Verhältnis zwischen einem anschaulich gegebenen Kontinuum und dem Zahlbegriff besser zu verstehen […], bei der *Zeit* als dem fundamentalsten Kontinuum: halten wir uns, um durchaus im Bereich des unmittelbar Gegebenen zu bleiben, an die *phänomenale* Zeit (im Gegensatz zur objektiven), an jene durchgängige Form meiner Bewußtseinserlebnisse, welche mir diese als in einem Ablauf aufeinanderfolgend erscheinen läßt. […] Um zunächst einmal überhaupt die Beziehung zur mathematischen Begriffswelt herstellen zu können, sei die ideelle Möglichkeit, in dieser [phänomenalen] Zeit ein streng punktuelles '*Jetzt*' zu setzen, sei die Aufweisbarkeit von Zeitpunkten zugegeben. Von je zwei verschiedenen Zeitpunkten ist dann immer der eine der *frühere*, der andere der *spätere*. (Weyl 1918 p. 67, Weyl's italics)

> In order to better understand the relation between an intuitively given continuum and the concept of number […] let us stick to *time* as the most

> fundamental continuum. And in order to remain thoroughly within the domain of the immediately given, let us adhere to *phenomenal* time (as opposed to objective time), i.e. to that persistent form of my experiences of consciousness by virtue of which they appear to me as succeeding each other in a sequence. […] In order to at all connect phenomenal time with the world of mathematical concepts, let us grant the ideal possibility that a strictly punctal *Now* can be posited within this [phenomenal] time and that time-points can be demonstrated. Given any two distinct time-points, one is the *earlier*, the other the *later*. (my translation)

This "*phenomenal* time" is obviously conceived as subjective (as opposed to so-called "objective time", which is presupposed as 'existent'), referring to a *sequence* (Ablauf) of internal conscious experiences, a kind of movement. The phenomenal character is due to intuition, i.e. Anschauung, from the German verb 'anschauen', 'to look at', having the same signification as Latin 'intueri'. Weyl considers (ibid. p. 66) the example of a pencil lying on the table in front of him *at which he is looking* to show that the pencil's position considered as a "mass-point" (Massenpunkt; ibid.) is a continuous function of time. He writes: "If the continuum of time is supposed to be represented by a variable 'running through' the real numbers, then, it seems, this determines how narrowly or broadly we have to conceive the concept of real number." (Soll sich das Zeitkontinuum durch eine die reellen Zahlen 'durchlaufende' Variable darstellen lassen, so, scheint es, ist damit gegeben, wie eng oder weit wir den Begriff der reellen Zahl zu fassen haben; ibid.) The phenomenally intuitive, sensuous starting-point is thus already fixated upon for making just such a correspondence with a destination, namely, "real number".

The kind of intuition appealed to as an incontrovertible phenomenal foundation is *immediate sensuous perception of something in the present*. The preferred sense is that of vision. Why should subjective, intuitive sense-perception, and time *as* a (linear) sequential movement of successive Nows be given such a self-evident lead role in searching for a foothold in phenomenal experience for the mathematical conception of a continuum of real numbers? Why is the subject-object split, i.e. the ostensibly self-evident dichotomy between an 'inside' and an 'outside' of consciousness (which goes along with the well-worn dichotomy in adversarial philosophical ‘positions’ between ‘idealism’ and ‘realism’) accepted without further ado as self-evident? Why is *sensuous* presence given priority? And why is *presence* given priority over absence? Why is time taken to be a kind of movement conceived *as* a linear succession of now-instants? None of these questions is raised by either Bell or Feferman; they precede that concerning whether it is legitimate to posit “punctal Nows” in order to make a bridge to the continuum of real numbers which is implicitly understood ‘self-evidently’ as a set of points.

Weyl himself ultimately discards as “nonsense” (Unsinn; ibid. p. 68) the attempt to establish a correspondence between the succession of moments in time intuitively ‘looked at’ from inside consciousness, and the real numbers, noting that “[t]he category of natural numbers probably can, but the continuum as it is given in intuition cannot, provide the foundation for a mathematical discipline. [...] already the concept of point in the continuum is lacking the necessary support in intuition for that.” (Wohl die Kategorie der natürlichen Zahlen, nicht aber das Kontinuum, wie es in der Anschauung gegeben ist, kann das Fundament einer mathematischen Disziplin abgeben. [...] bereits dem Begriff des Punktes im Kontinuum mangelt es dazu an der nötigen Stütze in der Anschauung. p. 68). Thus, according to Weyl, it is not justified by intuition to posit “punctal Nows”, which leaves “the concept of point in the continuum” without support. So much for the foundations of mathematical analysis.

He asks further, “Why is it that what is given to consciousness does not give itself as being pure and simple (as does, say, the logical being of concepts), but rather as an enduring and changing now-being — so that I can say: this is now — but now no longer?” (Worin liegt es, daß das Bewußtseins-Gegebene nicht als ein Sein schlechthin sich gibt (wie etwa das logische Sein der Begriffe), sondern als ein fortdauerndes und sich wandelndes Jetzt-sein — so daß ich sagen kann: Dies ist jetzt — doch jetzt nicht mehr? p. 69). This quotation shows that Weyl is presupposing i) an understanding of being *as* presence and ii) that what is given to consciousness is caught in a constant flow from ‘is’ to ‘is no longer’. If time itself is taken to be the ‘inner’ flow of now-moments abstracted from the flow of contents of consciousness, then it is conceived *as* a flowing succession of Nows in which one Now is and then is no longer. Hence there is a supposed matching between interior conscious, intuitive experience of the present and a flow of linear time that is, in particular, a *continuous* flow from being (is now) into non-being (is no longer) in which it is tacitly and unwittingly presupposed that ‘to be’ means ‘presently presencing’. Weyl does not note that an intuition of sequential flow is impossible without a ‘simultaneous’ consciousness of both a present instant and an instant that is “now no longer,” i.e. absent, past. The fixation of sensuous consciousness on the present Now has already implicitly been widened to include an absent Now.

It is telling that Weyl presumes that the *intuition* of “the category of natural numbers probably can [...] provide the foundation for a mathematical discipline”. In fact, he employs this intuitive category of the countably infinite natural numbers to present his own predicative definability as a solid basis for mathematical analysis, at least as far as 19th century analysis had gone (see Eldred 2009/2011 § 2.8.1 http://www.arte-fact.org/dgtlon_e.html#2.8.1), without having to invoke uncountable sets of real irrational numbers. (Note that I prefer to speak of matter-of-fact counting and countability rather than more erudite enumerating and enumerability.) The intuition of the natural numbers, however, is the (inner or outer or both?) conscious experience of counting, which itself is a succession, a kind of movement. This counting-experience can be taken to be simply the abstract (i.e. abstracted from contents of consciousness) counting of one moment after the other in which, starting with 1, a 1 is added successively in ongoing counting that, in principle, never ends. In this forward movement of counting, it is important to note that all the preceding numbers already gone through in the steady counting are also retained, although they are precisely *not* the ordinal number presently being counted. Although now *absent*, they are *retained* as having already been counted, which is itself a kind of *presence*.

Furthermore, although the counting experience of mortals is necessarily a finite one, it still can be conceived as going on ‘forever’ and thus as countably infinite, since it can be imagined that 'You can always add one more'. This has implications for the real numbers, too, each of which up to now (cf. however below *5 The mathematical continuum recast*) has been conceived as an endless sequence of discrete numerical (binary) digits and thus intuitively as an endless counting process. Hence, although each *single* real number is finitely or infinitely countable, i.e. rational or irrational, it is not possible to count the infinity of *all* real numbers taken together. These two different infinities, countable and uncountable, give rise to the so-called Continuum Hypothese (cf. Feferman 2011). But what does the set of all endless bit-strings have to do with the continuum conceived intuitively as the continuity of physical movement or geometrically as a straight line? Feferman does raise these questions when discussing different mathematical conceptions of the continuum. He writes e.g.,

> Appealing as the idea is of an arbitrary path through the binary tree, or an arbitrary sequence of 0s and 1s, the problem with this set-theoretical conception of the continuum is grasping the meaning of 'all' in the description of 2^N as consisting of all such sequences. (Feferman 2009 p. 14),

asking whether this 'all' can be considered as a "definite totality", i.e. whether it is predicable, sayable. In the end, what is definitely sayable (and therefore rational) is countable (rational in the mathematical sense). Because mathematics needs the real numbers for analysis, including, apparently, also the irrational real numbers, it presses on regardless, ignoring Weyl's qualms about finding an intuitive basis for the real continuum which even he cannot assuage. Indeed, at the end of *Das Kontinuum*, — and Bell (2000), Longo (1999), Feferman (2009) all cite this conclusion —, Weyl concludes there is no match between intuition and mathematical concepts:

> Dem Vorwurf gegenüber, daß von jenen logischen Prinzipien, die wir zur exakten Definition des Begriffs der reellen Zahl heranziehen müssen, in der Anschauung des Kontinuums nichts enthalten sei, haben wir uns Rechenschaft darüber gegeben, daß das im anschaulichen Kontinuum Aufzuweisende und die mathematische Begriffswelt einander so fremd sind, daß die Forderung des Sich-Deckens als absurd zurückgewiesen werden muß. Trotzdem sind jene abstrakten Schemata, welche uns die Mathematik liefert, erforderlich, um

> exakte Wissenschaft solcher Gegenstandsgebiete zu ermöglichen, in denen Kontinua eine Rolle spielen.
>
> To the criticism that there is nothing in the intuition of the continuum of those logical principles on which we must rely for the exact definition of the concept of real number, we have given the justification that the conceptual world of mathematics is so foreign to what is demonstrable in the intuitive continuum, that the demand for a perfect match between the two must be rejected as absurd. Nevertheless, those abstract schemata with which mathematics supplies us are required to enable an exact science of such domains of objects in which continua play a role. (Weyl 1918 p. 83, my translation)

This is a dispiriting conclusion, at least insofar as one could hope that mathematics provide an access to the world that accords with intuitive experience of it by immediately ‘looking at’ it, but perhaps Weyl painted himself into a corner — or rather enclosed himself in an interior cut off from the outside world — from the outset by relying on an intuition purportedly inside consciousness of a continuous flow of temporal instants. More generally it could — and must — be asked whether mathematics must necessarily do violence to the phenomena for the sake of “exact science” and whether this matters because, after all, exact science is judged ultimately by its effectivity in causally explaining the world with an eye to developing techniques of mastery over movement of all kinds. Perhaps the problem lies precisely with the *exactness* of mathematics. As I will attempt to show (see below *5 The mathematical continuum recast*), ultimately, the concept of irrational real number itself must be put into question, but first of all, the supposed dichotomy between an inside and outside of consciousness must be subjected to critical scrutiny.

2. Attempt to break out from inside consciousness: intersubjectivity

Mathematicians from the start must be ill at ease with a temporal intuition inside consciousness, for what could this inner intuition have to do with the external world supposedly outside consciousness? They seek so-called ‘objective’ mathematical truth with some solid ‘reality’. So perhaps Weyl simply had been misled by Husserl, who, before Weyl published *Das Kontinuum* in 1918, had held lectures in 1905 on the "phenomenology of inner temporal consciousness" (Husserl 1928) in which Husserl also assumes an "objective time" as self-evident which, however, is "bracketed off" for the phenomenological investigation (Husserl 1928 § 1 "Ausschaltung der objektiven Zeit"). Both Feferman and Longo attempt to break out from inside consciousness by invoking *intersubjectivity* as a kind of 'collective consciousness'. For instance, Feferman writes, "The objectivity of mathematics is a special case of intersubjective objectivity that is ubiquitous in social reality." (Feferman 2009 p. 4)

This invocation of "intersubjective objectivity" finds its philosophical support in John Searle: "[T]here are portions of the real world, objective facts in the world, that are only facts by human agreement. In a sense there are things that exist only because we believe them to exist. [...] things like money, property, governments, and marriages." (Searle 1995, p. 1 cited in ibid.) Note that the title of Searle's book reads *The Construction of Social Reality*, from which it can be inferred that in his view there is an objective reality constructed by subjects through agreement and convention. The example of money, such as a dollar bill, is revealing because the 'objective reality' of the piece of paper with "One dollar" printed on it is asserted to be money only by virtue of intersubjective agreement within some kind of collective consciousness. The subjects "believe" that the piece of paper is money with a certain value. For Feferman, this make-believe reality between the subjects suffices for him to assert,

> The objectivity of mathematics lies in its stability and coherence under repeated communication, critical scrutiny and expansion by many individuals often working independently of each other. Incoherent concepts, or ones which fail to withstand critical examination or lead to conflicting conclusions are eventually filtered out from mathematics. (Feferman 2009 ibid.)

Clearly, something more than a make-believe reality is implied for mathematics, for it is said to proceed by "critical examination" that leads to "conflicting conclusions" being "filtered out". This suggests some kind of Wittgensteinian language game, and Feferman does indeed invoke the example of the game of chess (ibid.) to underscore the "intersubjective objectivity" (ibid.) of social constructions: "...in the game of chess, it is not possible to force a checkmate with a king and two knights against a lone king." (ibid.) But aren't the rules of chess arbitrary, without necessary connection to naturally given phenomena that the mathematized sciences are interested in? Is the touchstone for "critical examination" merely a set of agreed rules for making mathematical statements in a kind of language game, or are there, and must there be, deeper roots in the experienced physical world? The first thesis of Feferman's "conceptual structuralism" reads:

> 1. The basic objects of mathematical thought exist only as mental conceptions, though the source of these conceptions lies in everyday experience in manifold ways, in the processes of counting, ordering, matching, combining, separating, and locating in space and time. (ibid. p. 3)

This thesis says that mathematical "objects" are (i) "only [...] mental conceptions", but (ii) they are rooted in certain everyday human practices that are (iii) located "in space and time". These "mental conceptions" are presumably "only" inside consciousness, and thus 'subjective objects', but gain an intersubjective objectivity through practices 'out there' in "space and time" which presumably are simply 'objective'. Notice the back-and-forth across the subject-object gulf, between an inside and outside, which is not altogether coherent, as signalled already by curious terms such as "intersubjective objectivity". Feferman finds support from his mathematician colleague, Giuseppe Longo:

> Discussing the continuum we have tried to describe how the mathematical intuition is built on our relation to the world, by 'these acts of experience ... within which we live as human beings' [Weyl 1918 p. 113?][2] On the basis of these life experiences, we propose descriptions and deduction, we make wagers, not arbitrary, but full of history and of intersubjectivity, of invariance within the plurality of experiences. (Longo 1999 typescript p. 7)

Once again, "intersubjectivity" is invoked as a kind 'objective' grounding. This intersubjective ground between the subjects then entices

2 The text of *Das Kontinuum* 1918 has only 87 pages. Longo's page reference presumably intends an English translation.

Longo to make a further step into 'objectivity', namely, from intersubjective objectivity to a Platonist ontology, as if this were the only place to go:

> For this reason, 'the mathematician must have the courage of his inner convictions; he will affirm that the mathematical structures have an existence independent of the mind that has conceived them; ... the Platonist hypothesis ... is ... the most natural and philosophically the most economical' [Thom 1990 p. 560]. Dana Scott more prudently said to this author: 'It does no harm'. (Longo 1999 typescript p. 6)

This "Platonist hypothesis" suffers from being a modern subjectivist (mis-)interpretation of Plato's ontology, for Plato does not have the problem of the gulf between an inside and outside of consciousness. Rather, he posits an existence of the ideas, i.e. the sights which beings present of themselves, in the "sky" (οὐρανός *Phaidros*) separate not from consciousness, but from the beings themselves. Such a separation of the ideas from the physical beings themselves, and locating them somewhere in a special place was criticized already by Aristotle. Moreover, Plato's ontology is a casting of the being of beings, i.e. their 'beingness' (οὐσία), and not merely some "economical" scientific "hypothesis". Philosophical thinking thinks differently from the prescriptions of (originally Cartesian) scientific method. Longo slips in a further unjustified assumption when he asserts "an existence independent of the mind", for what is the nature, i.e. the being, of the mind? Is it encapsulated inside subjective consciousness? Does it even make sense to posit any sort of "existence independent of the mind"? I shall come back to this question shortly.

The move that Feferman and Longo make from an inside of consciousness to an intersubjectivity raises the question as to the nature of the 'inter' between the subjective consciousnesses. Searle posits "agreement" and "convention" as this 'inter' but this is clearly not enough, for the agreement or convention refers also to certain things and practices in the world and becomes senseless without such reference. Can the mind be thought as 'inside' consciousness? If it can — and this is the fundamental positing of subjectivist metaphysics starting with ii) cogito ergo sum coupled with ii) res cogitans as subjective fundamentum absolutum vis-à-vis external iii) res extensa — then it inaugurates the multiple conundrums of a gulf between inside and outside consciousness in which philosophy, science and everyday thinking have become inextricably entangled in the Modern Age.

The way out of the antinomies of subjectivist metaphysics is to realize that the mind is always already outside, and there is no split at all, no inside and outside at all. The 'inter' of intersubjectivity conceived as a kind of collective consciousness is an ill-founded, flimsy, clumsy and superfluous bridge, for there could not be any agreements or conventions between the subjects without their always-already being outside, in the world. What stymies subjectivist metaphysics is that it does not know 'where' the mind is, nor that it has no 'where' at all. To risk some bald assertions that are backed up elsewhere (cf. Eldred 2012): In truth, the mind = time-clearing is all-encompassing; the only 'site' at which beings can present themselves at all AS beings to human awareness. This insight renders the problem of the 'inter' of intersubjectivity a pseudo-problem.

The mind is the open, three-dimensional time-clearing which is 'no-where', i.e. not a where, and also neither subjective nor objective, neither merely inside consciousness nor outside in the external world. Rather, the so-called external world is already inside the mind, and extended things (res extensa) can only take their places *within* the shared, all-encompassing mind, i.e. within the time-clearing, presencing and absencing themselves precisely here in this "pre-spatial" "time-space" (Heidegger 1962). The time-clearing is an all-encompassing, non-extended every-where and also a no-where, for it is not spatial at all, but rather that clearing within which extended things take their places, thus making space for space (cf. Eldred 2013). There is and can be no 'outside' the mind for it is humanly inconceivable. But I am running ahead of myself and will return to the nature of three-dimensional time below (*7 Clock-time and three-dimensional, ecstatic time*). Suffice it here to conclude for the moment that it is problematic to assume that stepping outside the inside of collective, intersubjective consciouness is a step into a reality external to and independent of the mind. Today's mathematicians are too quick to adduce an existence of "mathematical structures [...] independent of the mind" and are also thoroughly mistaken in labelling it a Platonist ontology.

So how does this help in finding a way out of the aporias of the ontological status of mathematical entities? Is there an intuitive basis in the phenomena of movement and time for a mathematical conception of the continuum? This question requires an interlude with Aristotle.

3. Time and continuum according to Aristotle

If, according to Weyl, Husserl invokes inner time as a sequential flow of sensuous, conscious experience, this interiority is absent from Aristotle's conception of time, which is famously a number (ἀριθμός) lifted off continuous movement, in particular, the regular, periodic motion of the celestial bodies ("Therefore it seems that time is the movement of the sphere,..." διὸ καὶ δοκεῖ ὁ χρόνος εἶναι ἡ τῆς σφαίρος κίνησις, *Phys*. IV 223b23):

> τοῦτο γάρ ἐστιν ὁ χρόνος, ἀριθμός κινήσεως κατὰ τὸ πρότερον καὶ ὕστερον. Οὐκ ἄρα κίνησις ὁ χρόνος, ἀλλ᾽ ᾗ ἀριθμὸν ἔχει ἡ κίνησις. (*Phys*. 219b1ff)
>
> This namely is time, the number of movement with respect to earlier and later. Time is therefore not movement but movement insofar as it has a number.

This determination is curious because *as* counted, number is discrete, whereas the movement from which it is lifted (ἀφαίρεσις) is continuous. Aristotle also says elsewhere explicitly that time is continuous: "... continuous, for instance, is line, surface and solid, as well as beside these time and place." (συνεχὲς δὲ οἷον γραμμή, επιφάνεια, σῶμα, ἐτι δε παρὰ ταῦτο χρόνος καὶ τόπος. *Cat*. 4b24). So it seems time is a continuous, periodic locomotion measured by some discrete arithmetic unit. But what about the incommensurable, irrational remainders when this measurement is made? Hence the same antinomy of incommensurability between the discrete and the continuous that haunts mathematics through the centuries is present already in Aristotle's famous casting of time itself as a number. Somehow each number counted in counting time is connected with its predecessor and successor number. Because of the ordinal numbering of counted time, it makes no sense to try to cut the present counted now off from earlier or later nows, so there is some sort of relation (πρός τι) or connection, and the earlier or later nows cannot be considered simply as 'non-existent', for otherwise the notion of succession would be lost. On the other hand, what justification is there to speak of a *continuous* 'time-line' when all there is in counting is a *discrete* succession (cf. below *5 The mathematical continuum recast*)?

What does Aristotle say about continuity? What is its special ontology, i.e. mode of being, or better, mode of presencing? (cf. *Phys*. V iii, Heidegger 2003, 'Excursus: General Orientation Regarding the Essence of Mathematics' pp. 69-82 and Eldred 2009/2011 § 2.1 http://www.arte-fact.org/dgtlon_e.html#2.1) Following preceding sections, perhaps it could be hazarded to ask for a special mode of presencing and absencing that characterizes continuity.

For Aristotle, continuity is one way in which (extended) physical beings (φύσει ὄντα) are (i.e. presence) together and, in particular, *hang together spatially* in the world or *move* through the world. These ways are 'together', 'separate', 'touching', 'between', 'succession', 'contiguous', 'continuous' (ἅμα, ἅπτεσθαι, μεταξύ, ἐφεξῆς, ἐχόμενον, συνεχές, respectively; *Phys*. V 226b18) and are built up successively from the simplest to the most complicated. The simplest is *togetherness*, when physical beings are at one *place* (τόπος), place being the envelope enveloping an extended physical being that enables it to presence *as* an extended physical being (non-physical beings not requiring places to presence). They *touch* when their extremities or limits (ἄκρα, πέρας, ὅρος) are together in the same place. *Succession* is when things come one after another, as with houses in a street when there is something in between that is not a house. *Contiguity* is when things hang together in the sense that their extremities touch each other, as when the outer walls of a row of houses in a street touch each other, so one can move from one house to the next without going through anything in between that is not a house. Finally, *continuity* is a strong contiguity in which the limits of the succeeding things not only touch, but are one and the same (ταὐτὸ καὶ ἓν τὸ ἑκατέρου πέρας, *Phys*. V

227a11) so that the things *hold* together and, as with contiguity, movement from one thing to the next remains within the same, as when a row of houses in a street is such that each pair of successive house shares the outer walls.

What implications do these different ways of being (i.e. presencing) together have for geometry and arithmetic? Aristotle conceives mathematical entities as abstracted from or lifted off (ἀφαίρεσις) extended physical beings. Geometrical entities result from lifting off the place that envelops a physical being as its contour, resulting in a geometric figure such as a line, surface or solid that is now placeless, but retains oriented position (θέσις). Geometrical figures of these three kinds are aesthetic in the sense of being perceptible by the senses which perceive their oriented position and, in particular, the oriented position of points (στιγμαί) on them in relation to each other. Arithmetic number arises from a different, more radical kind of abstraction from physical beings consisting in counting them, resulting in a definite number in a succession of numbers, each of which is discrete, i.e. distinguished from the others (διωρισμένον), and thus not only placeless (ἄτοπος), like geometrical figure, but also without orienting position (ἄθετος). Calculations can be done with these doubly abstracted numbers which do not rely on any aesthetic perception, whereas the manipulation of geometical figure requires an abstracted aesthetic perception, i.e. an intuition, of oriented figures in the imagination to 'see' what is happening.

Aristotle says that continuity applies to geometrical figure, i.e. to its three basic elements: line, surface and solid. How is the mathematical continuum to be conceived accordingly? It refers most primordially to a quality of the line hanging together tightly. This means that any line whatsoever can be bisected at any point of the line in such a way that the two resulting lines share an extremity, a limit. Conversely, points form the extremities of lines, and two lines can be joined together into a continuity when they share an end-point. Two lines intersect when they share a point of bisection. There can be *continuous movement* along a line because any point the movement passes through can be a shared point of bisection holding the two parts of the line tightly together. So the bisecting cut in fact does not separate the line into two distinguished parts, but rather leaves it whole by hinging it. This intuition is the basis for Dedekind's cut who, however, draws different conclusions (see below *4 Dedekind's attempt at grounding the real number continuum*).

What does this imply for *time*, which Aristotle claims is continuous, just as a geometrical line is? Time is the number lifted off the periodic, continuous motion of a celestial body along its orbit. The orbit describes a linear geometric figure of some kind such as a circle, an ellipse, a hyperbola or something more irregular. Aristotelean time is the number measuring along this orbit arithmetically on the basis of some unit. A number (ἀριθμός), however, is discrete, not continuous, taking its place in a *succession* of ordinal, counting numbers, so there is an ambiguity in Aristotle's conception of time as either a continuous, periodic movement *or* a discrete number measuring a continuous, periodic movement. As discrete, time is the number reached when counting in units along the line of orbit, leaving a remainder smaller than the counting-unit, which means that there are many (indeed, most) points on the line that are missed by counted time. For instance, the moon passing along its orbit in the night sky marks hours, or perhaps minutes, with its moving position at certain determinate points that can be determined by measuring the angle between the moon and a certain visible star (perhaps the sun) in a fixed unit. This measurement, of course, requires knowledge of the regular periodicity of the moon's orbit. On this Aristotelean account, time can be regarded as a discrete numerical approximation to a continuous, regular, periodic motion.

Taking a cue from this discussion of Aristotelean continuity and time, how could the mathematical continuum be redefined, or even recast, without simply invoking 'existence' of the real number continuum?

4. Dedekind's attempt at grounding the real number continuum

Before proceeding to propose a recasting of the continuum, it is instructive first to return to the seminal 1872 study by Richard Dedekind who is credited with putting the real number continuum onto a sound mathematical basis, not reliant on mere geometrical intuition that imagines infinitesimally small numbers, with his famous 'Dedekind cuts' in the rationals.

The problem of the mathematical continuum concerns the relationship between the geometric and the arithmetic, between figure and number, and has been with us ever since the Pythagorean discovery of irrational, incommensurable 'numbers', or rather, of irrational lengths of intervals, starting with the hypotenuse of a right-angled isosceles triangle with sides of unit length. When the problem of the motion of the celestial bodies was posed anew at the beginning of the Modern Age as a mathematical problem of how to *calculate* their motion, it became necessary to convert the geometric description of celestial bodies' motion into a calculative, arithmetic one consisting of soluble equations. This resulted in Newtonian mechanics with its familiar laws of motion that can be expressed in simple mathematical equations in terms of real numbers, i.e. of real number-points that are supposed to make up the real continuum. In this transposition of the geometric into the arithmetic, the geometric point in a figure was taken to correspond to a number, which is defined to be real because it pertains to 'real' motion of physical bodies. By virtue of the general nature of the mathematical equations they employ, the Newtonian laws of motion apply not only to celestial bodies, but to physical bodies in general.

Dedekind's contribution to the project to put the Newtonian and Leibnizian infinitesimal calculus onto a sound arithmetic basis, is to ground an arithmetic conception of the continuum. He proceeds in his 1872 study by first considering the system of rational numbers in § 1 with their elementary properties, in particular, their ordering, and then, in § 2, by comparing the rational numbers "with the points of a straight line" (Dedekind 1872 § 2) where, in particular, the ordering of points, i.e. their “positional relations” (Lagenbeziehungen, p. 14), is compared with the ordering of rational numbers. “§ 3 Continuity of the straight line” then introduces the famous Dedekind cut in order to develop a concept of continuity purely arithmetically. “The above comparison of the domain R of the rational numbers with the straight line has led to the knowledge of the gappiness, incompleteness or discontinuity of the former, whereas we ascribe to the straight line completeness, gaplessness or continuity (Die obige Vergleichung des Gebiets R der rationalen Zahlen mit einer Geraden hat zu der Erkenntnis der Lückenhaftigkeit, Unvollständigkeit oder Unstetigkeit geführt, während wir der Geraden Vollständigkeit, Lückenlosigkeit, oder Stetigkeit zuschreiben. Dedekind 1872 § 3 p. 17). This comparison, it must be said, is Dedekind’s basic *geometric* intuition.

Dedekind accordingly aims to fill in the well-known gaps between the rational numbers to attain completeness and hence continuity, i.e. gaplessness is here posited *as* equivalent to continuity, and irrational numbers are required to fill the gaps. He proceeds with his famous definition of the “cut” (Schnitt):

> Ich finde das Wesen der Stetigkeit [...] im folgenden Principe: ‘Zerfallen alle Puncte einer Geraden in zwei Classen von der Art, daß jeder Punct der ersten Classe links von jedem Punct der zweiten Classe liegt, so existirt ein und nur ein Punct, welche diese Eintheilung aller Puncte in zwei Classen, diese Zerschneidung der Gerade in zwei Stücke hervorbringt.’ (Dedekind 1872 § 3 p. 18)
>
> I find the essence of continuity [...] in the following principle: ‘If all the points of a staight line fall into two classes of the kind that each point in the first class lies to the left of every point of the second class, then there exists one and only one point that brings forth this division of all points into two classes, this cutting of the straight line into two pieces.’ (my translation)

This principle is to serve Dedekind to conceive the irrational numbers in the following § 4 entitled “Creation of the irrational numbers” (Schöpfung der irrationalen Zahlen). The guiding idea is to complete the rational numbers to a continuous domain through this “creation”. Dedekind first points out that any rational number divides the rationals into two classes in a way analogous to the cutting of a line into two, but, secondly, that there “exist also infinitely many cuts that are not brought forth by rational numbers” (auch unendlich viele Schnitte existiren, welche nicht durch rationale Zahlen hervorgebracht werden; ibid. § 4 p. 20). He provides an important illustration of this. The real numbers are then defined as comprising all the numbers that produce neat Dedekind cuts of the rationals. In this way, the rationals are filled up to form a complete, continuous ‘line’. Analogously, Dedekind discusses also how the gaps in real space could be filled in to produce a continuous space (cf. § 3 p. 19).

So what’s wrong with Dedekind’s line of reasoning, you may ask. Isn’t it perfectly obvious once you have been shown it? The first point to note is that Dedekind proposes to produce continuity by *cutting* into two, i.e. by bisecting, which, prima facie, is precisely the opposite of continuity. In his conception of the cut producing an irrational number, the dividing-point belongs to neither the left nor the right class, neither to the rationals less than or equal to the incisive irrational, nor to the rationals greater than or equal to the incisive irrational. In this sense the created irrational is truly cut off and unreachable from the rationals rather than joining them. Even with the rational cuts, the incisive rational number belongs to either one class (lesser than or equal, greater than or equal), but not to both, so the rational cut produces a separation, not a connection.

Recall from the above discussion of continuity according to Aristotle that, in the first place, continuity is a way in which physical entities in the world hang together tightly by sharing their extremities. They are not merely next to each other, touching each other in a contiguity. Dedekind’s basic conception of continuity of gaplessness is merely one of contiguity, or not even that, since each number is conceived for itself in a succession of numbers that *are as close as you like, but do not touch*. So here Dedekind is at loggerheads with Aristotle’s insight into the nature of continuity as a way in which things hang together rather than are separated from each other.[3]

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[3] Conversely, one could consider the physical conception of matter as being composed ultimately of particles such as molecules, atoms and a plethora of sub-atomic particles, suggesting that ‘physical reality’ is ultimately discrete, consisting of an unimaginably large collection of particles. Such a conception,

Furthermore, Dedekind's conception of the continuum of real numbers as a gapless sequence is based implicitly on the analogous conception of a line being composed of nothing other than points in "positional relations" to each other. Similarily, a solid would be composed of surfaces which, in turn, would be composed of lines which, in turn, would be composed of points. It is this conception of a mere manifold or set that is at work here which allows Dedekind, for instance, to conceive of space as a manifold of points that can be completed to gapless continuity. But a line is not just a set of points; rather, a line is composed of segments that hang together. This insight puts into question Dedekind's way of proceeding by "creating" irrational numbers to fill the gaps between the rationals to produce a "continuity" that is a mere collection of number-points.

To conclude this section, I note that Dedekind's comparison of the straight line with the rationals should have led him to consider how the rationals hang together, rather than to consider cuts and how the gaps between them could be filled by irrational number-points each existing for itself. If this is so, then it is questionable to posit or "create" irrational number-points at all, and another route must be taken. This conclusion compels me to take the decisive step of recasting the mathematical continuum which, of course, requires actually doing some mathematics.

5. The mathematical continuum recast

> Fassen wir den Mengenbegriff in dem präzisen Sinne, [...] so gewinnt die Behauptung, daß jedem Punkte einer Geraden (nach Wahl eines Anfangspunktes und einer Einheitsstrecke) als Maßzahl eine reelle Zahl [...] entspricht und umgekehrt, einen schwerwiegenden Inhalt. Sie stellt eine merkwürdige Verknüpfung her zwischen dem in der Raumanschauung Gegebenen und dem auf logisch-begrifflichem Wege

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of course, leaves aside consideration of how such particles hang together, sharing each other's extremities, perhaps through force-fields, rendering physical reality continuous. This latter conception is phenomenally closer to the mark, because physical reality is not merely a heap.

> Konstruierten. Offenbar aber fällt diese Aussage gänzlich aus dem Rahmen dessen heraus, was uns die Anschauung irgendwie über das Kontinuum lehrt und lehren kann; es handelt sich da nicht mehr um eine morphologische Beschreibung des in der Anschauung sich Darbietenden (das vor allem keine Menge diskreter Elemente, sondern ein fließendes Ganzes ist), vielmehr werden der unmittelbar gegebenen, ihrem Wesen nach inexakten Wirklichkeit exakte Wesen substruiert — ein Verfahren, das für alle exakte (physikalische) Wirklichkeitserkenntnis fundamental ist und durch welches allein die Mathematik Bedeutung für die Naturwissenschaft gewinnt. (Weyl 1918 end of Chap. 1 pp. 37f)

> If we conceive the concept of set in the precise sense, [...] then the assertion that to every point of a straight line (after choosing a starting-point and a unit length) there corresponds a real number as measuring-number, and conversely, gains a momentous content. It makes a remarkable connection between what is given in spatial intuition and what is constructed via logical concepts. Obviously, however, this statement lies completely outside the domain of what intuition somehow teaches, or can teach, us about the continuum; it is no longer a matter here of a morphological descriptioin of what is offered in intuition (which above all is not a set of discrete elements, but a fluid whole); rather, exact entities are constructed beneath immediately given reality, which is inherently inexact — a procedure fundamental for all exact knowledge of (physical) reality and through which alone mathematics gains importance for natural science. (my translation)

The quotation from Weyl chosen as motto for this section should serve as further stimulus for reconsidering the relationship between mathematics and the world as mediated by geometric intuition. Is mathematics' pretention to be the paradigmatic exact science par excellence allowing access to the world, despite its striking successes in effective mastery starting with Galileo, ultimately hubris?

But let me proceed by picking up the thread from the previous section: The correspondence Dedekind, among others, makes between the points on a line and number is a mismatch. *It is the line (γραμμή)* *and not the point (στιγμή) that corresponds to number (ἀριθμός),* which, in turn, is always countable and rational. The geometric point corresponds not to the unit, 1, but to zero, 0, because it is not points that generate geometric figures but, most primitively, lines or, more precisely, line intervals of whatever length, just as it is 1 that generates first the natural numbers N by successive counting (iteration), then the negative numbers and zero (which is the empty succession), and finally the rational numbers Q consisting of ratios of integers.

When seeking an arithmetic counterpart to the geometric continuum of the line, therefore, single number points are inappropriate. Instead, some kind of *rational (countable) intervals* must be involved. The first such correspondence is between a geometric interval taken as unit line and the arithmetic number, 1. With a view to connectivity, geometric points in a continuous line must be conceived in some way as the *end-points* of connected intervals composing, or at least *approximating*, the line in question. Even if the line is conceived *as* consisting solely of *rational* numbers, there is no limit to its rational divisibility, since between any two rational numbers, there are infinitely many rational numbers, and these rational numbers are *the* foothold for mathematics.[4] The rational end-points of connected closed line-intervals do not merely

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4 “[...] I [...] gained the firm conviction [...] that the *idea of iteration, of the natural number series, is an ultimate foundation of mathematical thinking*. [...] I see the greatness of mathematics precisely in the fact that in almost all its theorems what is essentially *infinite* is brought to a finite decision; this ‘infinity’ of mathematical problems, however, is based on the circumstance that *the infinite series of natural numbers and the concept of existence relating to them* constitute its foundation.” ([...] ich [...] gewann die feste Überzeugung [...], daß die *Vorstellung der Iteration, der natürlichen Zahlenreihe, ein letztes Fundament des mathematischen Denkens* ist. [...] Ich erblicke das Große der Mathematik gerade darin, daß in fast allen ihren Theoremen das seinem Wesen nach *Unendliche* zu endlicher Entscheidung gebracht wird; diese ‘Unendlichkeit’ der mathematischen Probleme beruht aber darauf, daß *die unendliche Reihe der natürlichen Zahlen und der auf sie bezügliche Existenzbegriff* ihre Grundlage bilden. Weyl 1918 p. 37) Weyl’s insight here should serve as a warning sign when attempting to pass beyond the rational numbers.

touch each other, rendering the line contiguous (ἐχόμενον), but are one and the same, rendering the line continuous (συνεχές) at that rational point. Correspondingly, closed rational intervals can be connected by sharing end-points as in [a, b] [b, c], where $a < b < c$ are all rational ($\in$ Q). Of course, if $a = b$ and/or $b = c$, the intervals would collapse to a single point and the meaning of continuity would become vacuous.

The Archimedean method of approximating a continuous line by ever smaller line-segments is well-known, serving as an intuitive basis for the development of the differential calculus from the outset. But how can this procedure be applied without invoking the 'existence' of irrational real numbers? It requires an alternative definition of 'reals'. To do this, I return to Cauchy's insight, summarized by Feferman as:

> Assuming the real number system R, Cauchy found a necessary and sufficient condition on arbitrary sequences (x) of real numbers in order for them to be convergent, namely that the $\lim_{n,m\to\infty} |x_n - x_m| = 0$ or, as we would put it since Weierstraß, that for any $k > 0$, there exists a p such that for all $n, m > p$, $|x_n - x_m| < 1/k$. (Feferman 2009 typescript p. 10),

but precisely *without* "assuming" the 'existence' of the real *number* system R. Instead, consider countably infinite sequences of closed rational intervals $\{[r_n - 1/n, r_n + 1/n] \mid r_n \in Q, n \in N]\}$ satisfying the Cauchy-convergence condition, namely, that for any $k \in N$ (the natural numbers), there exists $j \in N$ such that for all $n > m > j$, $|r_n - r_m| < 1/k$. This is equivalent to the condition that the combined length of the closed rational intervals $[r_n - 1/n, r_n + 1/n]$, $[r_m - 1/m, r_m + 1/m]$ becomes arbitrarily small for n, m sufficiently large, i.e. that for any $k \in N$, there exists $j > k \in N$ such that for all $n > m > j$, $|r_n - r_m| + 1/n + 1/m < 1/k$. This *Cauchy-convergence condition* (CC) implies that Cauchy-convergence depends only on the behaviour of the r_n for large n, since the length of the closed intervals approaches zero in any case.

5.1. A recast real continuum R

I define the (arithmetic) *continuum K* on the rational numbers Q to consist of all countably infinite sequences of closed rational intervals $\{<r_n, 1/n>\}$ defined as $\{[r_n - 1/n, r_n + 1/n] \mid r_n \in Q, n \in N]\}$, where

$[r_n - 1/n, r_n + 1/n] = \{x \mid x \in Q \text{ and } r_n - 1/n \leq x \leq r_n + 1/n\}$,

and the *real continuum R* to be the subset of K consisting of all *Cauchy-convergent sequences* of such closed rational intervals $\{<r_n, 1/n>\}$. Such a sequence may or may not have a rational limit $q \in Q$. If it does not, the Cauchy-convergent sequence $\{<r_n, 1/n>\}$ is said to be *irrational*. If it does, then the Cauchy-convergent sequence $\{<r_n, 1/n>\}$ is said to be *rational* and there is a rational limit $q \in Q$ satisfying the usual convergence condition that for any $k \in N$, there exists $j > k \in N$ such that for all $n > j$, $|r_n - q| + 1/n < 1/k$. Any rational number $q \in Q$ can be represented canonically by the rational Cauchy-convergent sequence $\{<r_n, 1/n> \mid r_n = q \text{ for all } n \in N\}$.

A diagonal argument shows that there are uncountably many irrational Cauchy-convergent sequences and also uncountably many rational Cauchy-convergent sequences to a given rational limit $q \in Q$. So there are also uncountably many ‘real’, countably infinite, sequences of rational intervals all told in R.

5.2. Arithmetic operations on R

The normal arithmetic operations can be carried out on such Cauchy-convergent sequences of rationals element-wise, e.g. addition:

$\{<r_n, 1/n>\} + \{<s_n, 1/n>\} = \{<r_n + s_n, 1/n>\}$.

Division is possible as long as the denominator Cauchy-convergent sequence does not converge on 0. Consider the division of two Cauchy-convergent sequences $\{<s_n, 1/n>\}/\{<r_n, 1/n>\} = \{<s_n/r_n, 1/n> \mid r_n \neq 0\}$, which is defined and Cauchy-convergent if $\{<r_n, 1/n>\}$ does not converge on 0, i.e. there is $k \in N$ such that for all $j > 4k \in N$ there is an $n^* > j$ with $|r_{n^*} - 0| + 1/n^* \geq 1/k$.

Rearranging, we have $|r_{n^*}| \geq 1/k - 1/n^* > 1/k - 1/4k = 3/4k > 0$ (A), so there are infinitely many non-zero r_n to form an infinite sequence of s_n/r_n. Moreover, since $\{<r_n, 1/n>\}$ is Cauchy-convergent, for this $k \in N$ by (CC) there is $h > 4k \in N$ such that for all $p > m > h > 4k$ $|r_p - r_m| + 1/p + 1/m < 1/k$. Choose $m = n^* > h > 4k \in N$. Then, for all $p > n^* > h > 4k$ it holds:

$|r_p - r_{n^*}| < 1/k - 1/p - 1/n^* < 1/k - 1/4k - 1/4k = 1/2k$ (B)

There are three cases to consider:

i) if $|r_p| \geq |r_{n*}|$, then $|r_p| > 0$, since $|r_{n*}| > 1/2k > 0$

ii) if $|r_p| < |r_{n*}|$ and $|r_{n*}| \geq 1/k$, then:

$|r_p| = |r_{n*} - (r_{n*} - r_p)| = |r_{n*}| - |r_{n*} - r_p| > 1/k - 1/2k = 1/2k > 0$, employing (B) in the penultimate step and

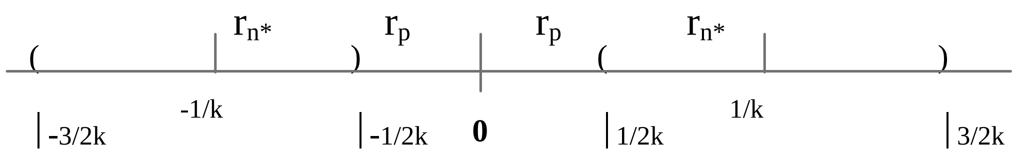

Diagram for case iii)

iii) if $|r_p| < |r_{n*}|$ and $|r_{n*}| < 1/k$, then:

$3/4k < |r_{n*}| < 1/k$ by (A) and

$|r_p| = 1/k - |1/k - r_{n*}| - |r_{n*} - r_p| > 1/k - (1/k - 3/4k) - 1/2k = 1/4k$, employing (A) and (B) in the penultimate step.

So, in any case, from a certain point onward ($p > n*$), all the r_p are non-zero and so can be a divisor in s_p/r_p. to form the infinite quotient Cauchy-convergent sequence, $\{<s_n/r_n, 1/n> \mid r_n \neq 0 \text{ for all } n > n*\}$.

5.3. Continuity of functions on R

Now consider the family of functions $F: R \rightarrow R$, $F(\{<r_n, 1/n>\}) = \{<f(r_n), 1/n>\}$, where f is a rational function $f: Q \rightarrow Q$. F (or f) is said to be *continuous* at $q \in Q$ if, for any $\{<r_n, 1/n>\}$ converging on q, $\{<f(r_n), 1/n>\}$ is also rationally convergent, converging on f(q). Otherwise F (or f) has a *discontinuity* at q.

5.4. Differentiability of continuous functions on R

For F continuous at $q \in Q$, consider the closed rational intervals $\{<r_n, 1/n>\} = \{[r_n - 1/n, r_n + 1/n] \mid r_n \in Q, n \in N]\}$ converging on q and also the closed rational intervals:

$\{<f(r_n), 1/n>\} = \{[f(r_n) - 1/n, f(r_n) + 1/n] \mid r_n \in Q, n \in N]\}$

converging on f(q). The rational line-segment passing through the end-points of these closed rational intervals has the rational equation:

$s(x) = f(r_n - 1/n) + (x - r_n + 1/n)\{f(r_n + 1/n) - f(r_n - 1/n)\}/2n, \qquad x \in Q$

and its slope is

$\{f(r_n + 1/n) - f(r_n - 1/n)\}/2n$.

F (or f) is said to be *differentiable* at $q \in Q$ if it is continuous at q — i.e. $\lim_{n\to\infty} \{\langle r_n, 1/n\rangle\} = q$ and $\lim_{n\to\infty} \{\langle f(r_n), 1/n\rangle\} = f(q)$ — *and also* the slope sequence, $\delta f(r_n)/\delta(1/n) = \{\langle (f(r_n + 1/n) - f(r_n - 1/n))/2n, 1/n\rangle\}$, is Cauchy-convergent, either rationally or irrationally. Only in the former case can a differential $\delta f(r_n)/\delta(1/n)$ be calculated as a definite rational number at q.

What implications does this conception of the reals as infinite Cauchy-convergent sequences of closed rational intervals, rather than definite numbers, have for the mathematization of movement and time?

6. Indeterminacy of movement and time

The Aristotelean conception of time as a number lifted off a regular, periodic motion according to the succession of earlier and later need not be restricted to the orbital motions of celestial bodies (that can be measured, say, by a sun dial or some more accurate instrument, such as a sextant), but applies equally well to the regular, periodic motion of artificial, mechanical devices of all kinds that ape the regular periodicity of celestial motion. In this way, the counting unit can be made ever smaller and thus the counting of time ever more accurate, right down to, say, counting the oscillations of a quartz crystal or a caesium atom. This is *clock-time* as investigated in extenso in Heidegger's *Sein und Zeit* (1927), to which I shall return shortly. No matter how accurate clocks (today called 'chronometers', literally 'time-measurers', by physicists) become, they remain a discrete counting with an incommensurable remainder that can never be counted. In this sense, all clocks are essentially digital (since any number whatsoever can be expressed as a binary number, i.e. as a bit-string). Modern science confuses the primordial phenomenon of time itself with measurable, countable clock-time and deludes itself that with clocks of ever greater accuracy it were approaching the 'nature' of time itself. (This is apparent in contemporary attempts at theories of time on a quantum level; cf. Eldred 2009/2011 § 7.3.3 http://www.arte-fact.org/dgtlon_e.html#7.3.3) The nature of clock-time can be captured mathematically by restricting the time variable, t, to the (countably infinite) rational numbers Q.

The mathematization of movement and time consists in plotting movement against time by a mathematical function,[5] the first kind of movement being locomotion, i.e. the change of position, which is now plotted against rational clock-time, t. Aristotelean change of place, κίνησις κατὰ τόπον, is *geometrized* to change of position, κίνησις κατὰ θέσιν, and then, in a further abstraction, it is *arithmetized* to change of number, κίνησις κατὰ ἀριθμόν. It is with this second abstraction to so-called Cartesian co-ordinates that a sleight of hand takes place by positing the 'existence' of the continuum of real numbers, so that both arithmetized motion and arithmetized time are posited *as* real, continuous variables in such a way that continuous position x can be written *as* a function of the assumed continuous real variable, t, thus: x = f(t). *'As'* is italicized twice in the preceding sentence because here an

[5] Cf. "Historically, the concept of function has a twofold root. To it led *firstly*, the 'naturally given dependencies' ruling in the material world which, on the one hand, consist in the fact that circumstances and qualities of real things are variable over *time*, the independent variable par excellence, and, on the other hand, in the causal connections between cause and effect. A *second*, completely independent root resides in the arithmetic-algebraic operations. [...] The point where these two initially mutually completely alien sources of the concept of function come into contact is the concept of *natural law*: its essence consists precisely in the fact that in the natural law a naturally given dependency is represented as a function constructed in a purely conceptual-arithmetic way. Galileo's laws of falling bodies are the first great example." (Historisch hat der Funktionsbegriff eine doppelte Wurzel. Zu ihm führten *erstens* die in der materiellen Welt herrschenden 'naturgegebenen Abhängigkeiten', die einerseits darin bestehen, daß Zustände und Beschaffenheiten realer Dinge veränderlich sind in der *Zeit*, der unabhängigen Veränderlichen κατ᾽ ἐξοχὴν, anderseits in den kausalen Zusammenhängen zwischen Ursache und Wirkung. Eine *zweite*, von dieser ganz unabhängige Wurzel liegt in den arithmetisch-algebraischen Operationen. [...] Die Stelle, an der die beiden einander zunächst ganz fremden Quellen des Funktionsbegriffs in Beziehung zueinander treten, ist der Begriff des *Naturgesetzes*: sein Wesen besteht eben darin, daß im Naturgesetz eine naturgegebene Abhängigkeit als eine auf rein begrifflich-arithmetische Weise konstruierte Funktion dargestellt wird. Galileis Fallgesetze sind das erste große Beispiel. Weyl 1918 pp. 34f). This is a clear statement of the will to power behind the drive toward mathematization, not only in physics.

hermeneutic As is at work, casting motion and time *as* such-and-such in such a way as to enable mathematization. It is the will to mathematization for the sake of calculating — thus predicting, controlling — movement that from the outset dictates such an hermeneutic casting.

If, however, clock-time is irremediably countably rational, despite all refinement of the counting unit reflected in the denominator of the number, then a continuous real time-continuum cannot be assumed to ‘exist’ (i.e. to presence in the present) at all. Similarly, any continuous motion (or indeed any movement/change) at all can only be mathematized as a rational function without assuming the ‘existence’ of the real number continuum along which position (or, more generally, dynamic state) is plotted in a graph of the kind $x = f(t)$ where x, t are rational numbers, and not elements of an assumed real continuum.

Instead, as shown in the preceding section, the reals have to be conceived as Cauchy-convergent sequences of closed rational intervals that *may* close in on and thus surround some rational limit, or *may not*, i.e. the sequential closed rational intervals become as close as you like to each other without, however, ever approaching a numerical, rational limit. Instead, they hover forever in the *irrational indeterminacy* of a *multiple presencing*; there is *at most* only ever an approximation, a nearing, toward a rational state $x = f(t_0)$ as the sequence of rational time intervals approaches its rational clock-time limit t_0. For this to happen, a rational Cauchy-convergent time sequence $\{<t_n, 1/n> \mid t_n \in Q, n \in N\}$ converging on t_0 must be assumed for which it also holds that the Cauchy-convergent sequence of rational intervals $\{<f(t_n), 1/n> \mid t_n \in Q, n \in N\}$ also approaches a rational limit, $f(t_0)$. There ‘exist’, however, also uncountably many irrationally Cauchy-convergent time sequences that have no rational limit t_0. Correspondingly, there are uncountably many irrationally Cauchy-convergent sequences

$$\{<f(t_n), 1/n> \mid t_n \in Q, n \in N\}$$

lacking any rational limit and hence also uncountably many corresponding irrationally hovering dynamic states.

The *continuum of real time, T*, consisting of all Cauchy-convergent sequences of closed rational time-intervals, can never be determined as a continuum of definite numbers, whether rational or irrational. Why not? Because irrational numbers themselves do not 'exist', i.e. do not uniquely presence in the present, and there are uncountably many *irrationally* Cauchy-convergent sequences of closed rational time-intervals. Real time $t \in T$ itself hovers always as the multiple presencing of an infinite sequence $\{<t_n, 1/n>\}$ of Cauchy-convergent rational intervals, each containing a countable infinity of earlier and later rational time-points, no matter how small it is. Correspondingly, any function of real time $t \in T$, $F(t) = \{<f(t_n), 1/n>\}$, has infinite sequences of position (or dynamic state) $f(t_n)$ that are *irrationally* Cauchy-convergent containing countably many infinite positions (or dynamic states) both earlier and later. Only rational time-points measured by some more or less accurate clock can be determined; there are no irrational time-points but only ever smaller closed, rational time-intervals.

There is an essential hovering indeterminacy for any mathematized description of movement consisting in the dynamic state 'square' with 'sides' $<f(t_n), 1/n>$ by $<t_n, 1/n>$. This indeterminacy resides in the nature of mathematization itself, prior to any experimental result that physics may present, such as so-called quantum indeterminacy, which itself is a result reliant on inadmissibly positing from the outset, i.e. a priori, the existence of a real time continuum T of real *numbers* rather than Cauchy-convergent infinite *sequences* of closed, rational time intervals.

Likewise, the calculability of the *rate of change* of position (or dynamic state), also known as 'velocity', is subject to an essential indeterminacy residing in the nature of differentiability itself. As shown in the preceding section, differentiability at some rational point (here: t_0) depends upon the continuity of f(t) at t_0 and the Cauchy-convergence of the differential sequence,

$\delta f(t_n)/\delta(1/n) = \{<(f(t_n + 1/n) - f(t_n - 1/n))/2n, 1/n>\}$.

Hence, given a rational clock-time point, t_0, for which the dynamic state $F(t) = \{<f(t_n), 1/n>\}$ converges on $f(t_0)$ as $\{<t_n, 1/n>\}$ converges on t_0, the velocity of change may be either rationally or irrationally convergent but, in any case, is indeterminate since defined only by everdecreasing rational intervals. Thus, position and velocity remain forever in an indeterminacy toward each other, and it is not possible to determine even one of these variables definitely, since *both* are given only hoveringly within rational intervals corresponding to rational time intervals. This a priori mathematical result goes beyond that of Heisenberg indeterminacy for which at least one of the variables could be definitely determined. It is more radical in the double sense that i) it is mathematical, relying only on considerations concerning the mathematization of 'irrational' movement and time, and thus prior to experience and, consequently, ii) it applies not merely to the so-called quantum scale of the sub-atomic, but to anything at all that moves, i.e. to physical beings in general on both micro and macro scales.

The claim that the irrational real numbers do not exist means that as definite numbers there are only the countable, rational end-points of intervals enclosing countably many rational numbers and also countably-infinite sequences of such intervals that do not converge rationally at all, but get as small as you like nonetheless by virtue of Cauchy-convergence. A limiting end-point remains forever withheld in absence, and cannot even be named (predicated), whereas for rationally convergent sequences, the limiting value remaining forever withheld in absence is at least a definite rational number. This insight into the non-existence of real numbers has now been transferred to mathematized movement and (rational, countable, clock-)time to see more clearly the indeterminacy inherent in *all* physical movement.

7. Clock-time and three-dimensional, ecstatic time

Rational, countable clock-time is counted along the (time-)line of ordered, counted rational numbers which is absolutely indispensable for conceiving *efficient causality* and its mathematization in equations of movement of all kinds. Ideally, and starting with locomotion, position is to be determined precisely and made calculable by an equation in rational, countable clock-time. This is the only kind of time that interests physics because its calculations always rely ultimately on rational, numeric data and calculations resulting ultimately in concrete, rational, numerical results such as a determination of, say, the distance of a galaxy from our solar system or the so-called gravitational constant, even if such physical magnitudes are conceived *as* continuous real numbers within the mathematical theory. So physicists pretend that irrational real numbers 'exist' in a continuum for the sake of their usefulness in computability. Their existence is then asserted to be simply an intersubjective agreement of convenience (cf. above *2 Attempt to break out from inside consciousness: intersubjectivity*).

But even a conventional philosopher of mathematics such as Longo unwittingly invokes more than merely linear time when he writes, "...for Weyl the temporal continuum does not have points, the instants are merely 'transitions', the present is only possible due to the simultaneous perception of the past and of the future," (Longo 1999 typescript p. 2) or "...time itself *is* the simultaneous perception of the past, the present and the future. The present time that is not there anymore, it is past, or that is not there yet, it is future, and that we only understand when inserted in the whole of time or within a segment of time." (Longo 1999 typescript p. 5, Longo's italics). In these two quotations, i) the existence of *points* in time is denied in favour of transitionary time *intervals* in which not only the present, but also the past and future, are necessarily present, ii) human perception itself is asserted to be a strangely "simultaneous" one of all three temporal dimensions, and iii) an intimate relation between time and human perception itself is indicated. These hints by Longo in fact have consequences for breaking with the conception of the real time continuum of real *numbers* and also with the conception of rationally countable clock-time, for neither conception suffices for conceiving coherently a "simultaneous" presencing of all three temporal dimensions.

For this reason, in *Sein und Zeit* (1927), Heidegger refers to clock-time as a "vulgar" time relative to the "primordial" temporality of Dasein's Da (cf. Eldred 2011) from which linear, successive clock-time is only a derivative modification. Human being (Dasein) is exposed most originarily to the three-dimensional time-clearing (cf. Eldred 2013), and this is its *mind* through which it perceives the world (cf. Eldred 2012). In particular, the human mind sees something in motion 'simultaneously' where it is, where it was and where it will be. This perceiving mind is not encapsulated somehow inside, but is always already standing ‘ec-statically’ out there, exposed to and encompassing the temporal dimensions of present, future and past in the unified time-clearing.

Hence it makes no sense to want to distinguish between a ‘subjective time’ of consciousness and an ‘objective time’ out there in the ‘external’ world entirely independent of consciousness. Rather, the time-clearing itself needs human being for itself to be that opening for the presencing and absencing of all that presents and absents itself *as* presents and absents (i.e. occurrents). Without human being there is no *As* whatsoever and it is vacuous for human being to try to imagine and postulate what is ‘there’ external to any possibility whatsoever of its ever presenting itself. There is no, nor can there be, any outside to the time-clearing. Such an outside is literally inconceivable for the human mind and hence senseless, vacuous, a figment of thinking imagination that is confused about what it is imagining.

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